BUFFON NEEDLE LANDS IN ϵ-NEIGHBORHOOD OF A 1-DIMENSIONAL SIERPINSKI GASKET WITH PROBABILITY AT MOST $|\log \epsilon|^{-c}$

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ABSTRACT. In recent years, relatively sharp quantitative results in the spirit of the Besicovitch projection theorem have been obtained for self-similar sets by studying the L^p norms of the "projection multiplicity" functions, f_θ, where $f_\theta(x)$ is the number of connected components of the partial fractal set that orthogonally project in the θ direction to cover x. In [4], it was shown that n-th partial 4-corner Cantor set with self-similar scaling factor 1/4 decays in Favard length at least as fast as $\frac{C}{n^p}$, for $p < 1/6$. In [1], this same estimate was proved for the 1-dimensional Sierpinski gasket for some $p > 0$. A few observations were needed to adapt the approach of [4] to the gasket: we sketch them here. We also formulate a result about all self-similar sets of dimension 1.

1. DEFINITIONS AND RESULT

Let $E \subset \mathbb{C}$, and let proj_θ denote orthogonal projection onto the line having angle θ with the real axis. The **average projected length** or **Favard length** of E, $\text{Fav}(E)$, is given by

$$\text{Fav}(E) = \frac{1}{\pi}\int_0^\pi |\text{proj}_\theta(E)|d\,\theta.$$

For bounded sets, Favard length is also called **Buffon needle probability**, since up to a normalization constant, it is the likelihood that a long needle dropped with independent, uniformly distributed orientation and distance from the origin will intersect the set somewhere.

$B(z_0, r) := \{z \in \mathbb{C} : |z - z_0| < r\}$. For $\alpha \in \{-1, 0, 1\}^n$ let

$$z_\alpha := \sum_{k=1}^{n} (\frac{1}{3})^k e^{i\pi[\frac{1}{2}+\frac{2}{3}\alpha_k]}, \;\; \mathcal{G}_n := \bigcup_{\alpha\in\{-1,0,1\}^n} B(z_\alpha, 3^{-n}).$$

This set is our approximation of a partial Sierpinski gasket; it is strictly larger. We may still speak of the approximating discs as "Sierpinski triangles."

The main result:

Theorem 1. *$Fav(\mathcal{G}_n) \le \frac{C}{n^c}$, $c > 0$.*

Set $\mathcal{G}_n$ is 3^{-n} approximation to Besicovitch irregular set (see [2] for definition) called Sierpinski gasket. Recently one detects a considerable interest in estimating the Favard length of such ϵ-neighborhoods of Besicovitch irregular sets, see [5], [6], [4], [3]. In [5] a random model of such Cantor set is considered and estimate $\asymp \frac{1}{n}$ is proved. But for non-random self-similar sets the estimates of [5] are more in terms of $\frac{1}{\log \cdots \log n}$ (number of logarithms depending on n) and more suitable for general class of "quantitatively Besicovitch irregular sets" treated in [6].

Let $f_{n,\theta} := \frac{1}{2}\nu_n * 3^n \chi_{[-3^{-n},3^{-n}]}$, where

$$\nu_n := *_{k=1}^n \widetilde{\nu}_k \text{ and } \widetilde{\nu}_k := \frac{1}{3}[\delta_{3^{-k}cos(\pi/2-\theta)} + \delta_{3^{-k}cos(-\pi/6-\theta)} + \delta_{3^{-k}cos(7\pi/6-\theta)}].$$

For $K > 0$, let $A_K := A_{K,n,\theta} := \{x : f_{n,\theta} \ge K\}$. Let $\mathcal{L}_{\theta,n} := \text{proj}_\theta(\mathcal{G}_n) = A_{1,n,\theta}$. For our result, some maximal versions of these are needed:

$$f^*_{N,\theta} := max_{n\le N} f_{n,\theta}, \;\; A^*_K := A^*_{K,n,\theta} := \{x : f^*_{n,\theta} \ge K\}.$$

Also, let $E := E_N := \{\theta : |A^*_K| \le K^{-3}\}$ for $K = N^{\epsilon_0}$, ϵ_0.

Later, we will jump to the Fourier side, where the function

$$\varphi_\theta(x) := \frac{1}{3}[e^{-i\cos(\pi/2-\theta)} + e^{-i\cos(-\pi/6-\theta)} + e^{-i\cos(7\pi/6-\theta)}]$$

plays the central role: $\widehat{\nu_n}(x) = \prod_{k=1}^n \varphi_\theta(3^{-k}x)$.

2. General philosophy

Fix θ. If the mass of $f_{n,\theta}$ is concentrated on a small set, then $||f_{n,\theta}||_p$ should be large for $p > 1$ - and vice versa. $1 = \int f \le ||f_{n,\theta}||_p ||\chi_{\mathcal{L}_{\theta,n}}||_q$, so $m(\mathcal{L}_{\theta,n}) \ge ||f||_p^{-q}$, a decent estimate. The other basic estimate is not so sharp:

$$m(\mathcal{L}_{\theta,N}) \le 1 - (K-1)m(A_{K,N,\theta}) \tag{2.1}$$

However, a combinatorial self-similarity argument of [4] and revisited in [1] shows that for the Favard length problem, it bootstraps well under further iterations of the similarity maps:

Theorem 2. *If $\theta \notin E_N$, then $|\mathcal{L}_{\theta,NK^3}| \le \frac{C}{K}$.*

Note that the maximal version f^*_N is used here. A stack of K triangles at stage n generally accounts for more stacking per step the smaller n is. For fixed $x \in A^*_{K,N,\theta}$,

the above theorem considers the smallest n such that $x \in A_{K,n,\theta}$, and uses self-similarity and the Hardy-Littlewood theorem to prove its claim by successively refining an estimate in the spirit of (2.1). Of course, now Theorem 1 follows from the following:

Theorem 3. *Let $\epsilon_0 < 1/11$. Then for $N >> 1$, $|E| < N^{-\epsilon_0}$.*

It turns out that L^2 theory on the Fourier side is of great use here. It is proved in [4], [1]:

Theorem 4. *For all $\theta \in E_N$ and for all $n \le N$, $||f_{n,\theta}||_{L^2}^2 \le CK$.*

One can then take small sample integrals on the Fourier side and look for lower bounds as well. Let $K = N^{\epsilon_0}$, and let $m = 2\epsilon_0 \log_3 N$. Theorem 4 easily implies the existence of $\tilde{E} \subset E$ such that $|\tilde{E}| > |E/2|$ and number n, $N/4 < n < N/2$, such that for all $\theta \in \tilde{E}$,

$$\int_{3^{n-m}}^{3^n} \prod_{k=0}^{n} |\varphi_\theta(3^{-k}x)|^2 dx \le \frac{2CKm}{N} \le 2\epsilon_0 N^{\epsilon_0 - 1} \log N.$$

Number n does not depend on θ; n can be chosen to satisfy the estimate in the average over $\theta \in E$, and then one chooses $\tilde{E}$. Let $I := [3^{n-m}, 3^n]$.

Now the main result amounts to this (with absolute constant A large enough):

Theorem 5.

$$\theta \in \tilde{E} : \int_I \prod_{k=0}^{n} |\varphi_\theta(3^{-k}x)|^2 dx \ge c3^{m-2\cdot Am} = cN^{-2\epsilon_0(2A-1)}.$$

The result: $2\epsilon_0 \log N \ge N^{1-\epsilon_0(4A-1)}$, i.e., $N \le N^*$. Now we sketch the proof of Theorem 5. We split up the product into two parts: high and low-frequency: $P_{1,\theta}(z) = \prod_{k=0}^{n-m-1} \varphi_\theta(3^{-k}z)$, $P_{2,\theta}(z) = \prod_{k=n-m}^{n} \varphi_\theta(3^{-k}z)$.

Theorem 6. *For all $\theta \in E$, $\int_I |P_{1,\theta}|^2 \, dx \ge C\, 3^m$.*

Low frequency terms do not have as much regularity, so we must control the damage caused by the **set of small values**, $SSV(\theta) := \{x \in I : |P_2(x)| \le 3^{-\ell}\}$, $\ell = \alpha\, m$ with sufficiently large constant α. In the next result we claim the existence of $\mathcal{E} \subset \tilde{E}$, $|\mathcal{E}| > |\tilde{E}/2|$ with the following property:

Theorem 7.

$$\int_{\tilde{E}} \int_{SSV(\theta)} |P_{1,\theta}(x)|^2 dx\, d\theta \le 3^{2m-\ell/2} \Rightarrow \forall \theta \in \mathcal{E} \int_{SSV(\theta)} |P_{1,\theta}(x)|^2 dx\, d\theta \le c\, K\, 3^{2m-\ell/2}\,.$$

Then Theorems 6 and 7 give Theorem 5.

3. Locating zeros of P_2

We can consider $\Phi(x,y)=1+e^{ix}+e^{iy}$. The key observations are

$$|\Phi(x,y)|^2\ge a(|4\cos^2 x-1|^2+|4\cos^2 y-1|^2)\,,\quad \frac{\sin 3x}{\sin x}=4\cos^2 x-1\,.$$

Changing variable we can replace $3\varphi_\theta(x)$ by $\phi_t(x)=\Phi(x,tx)$. Consider $P_{2,t}(x):=\prod_{k=n-m}^n\frac13\phi_t(3^{-k}x)$, $P_{1,t}(x):=\prod_{k=0}^{n-m}\frac13\phi_t(3^{-k}x)$. We need $SSV(t):=\{x\in I: |P_{2,t}(x)|\le 3^{-\ell}\}$. One can easily imagine it if one considers $\Omega:=\{(x,y)\in[0,2\pi]^2: |\mathcal{P}(x,y)|:=|\prod_{k=0}^m\Phi(3^kx,3^ky)|\le 3^{m-\ell}\}$. Moreover, (using that if $x\in SSV(t)$ then $3^{-n}x\ge 3^{-m}$, and using $xdxdt=dxdy$) we change variable in the next integral:

$$\int_{\tilde E}\int_{SSV(t)}|P_{1,t}(x)|^2\,dxdt=3^{-2n+2m}\cdot 3^n\int_{\tilde E}\int_{3^{-n}SSV(t)}|\prod_{k=m}^n\Phi(3^kx,3^ktx)|^2\,dxdt\le$$

$$3^{-n+3m}\int_\Omega|\prod_{k=m}^n\Phi(3^kx,3^ky)|^2\,dxdy\,.$$

Now notice that by our key observations $\Omega\subset\{(x,y)\in[0,2\pi]^2: |\sin 3^{m+1}x|^2+|\sin 3^{m+1}y|^2\le a^{-m}3^{2m-2\ell}\le 3^{-\ell}\}$. The latter set $\mathcal{Q}$ is the union of $4\cdot 3^{2m+2}$ squares Q of size $3^{-m-\ell/2}\times 3^{-m-\ell/2}$. Fix such a Q and estimate

$$\int_Q|\prod_{k=m}^n\Phi(3^kx,3^ky)|^2\,dxdy\le 3^\ell\int_Q|\prod_{k=m+\ell/2}^n\Phi(3^kx,3^ky)|^2\,dxdy\le$$

$$3^\ell\cdot(3^{-m-\ell/2})^2\int_{[0,2\pi]^2}|\prod_{k=0}^{n-m-\ell/2}\Phi(3^kx,3^ky)|^2\,dxdy\le 3^\ell\cdot(3^{-m-\ell/2})^2\cdot 3^{n-m-\ell/2}=3^{-2m}\cdot 3^{n-m-\ell/2}\,.$$

Therefore, taking into account the number of squares Q in $\mathcal{Q}$ and the previous estimates we get

$$\int_E\int_{SSV(t)}|P_{1,t}(x)|^2\,dxdt\le 3^{2m-\ell/2}\,.$$

Theorem 7 is proved.

To prove Theorem 6 we need the following simple lemma.

Lemma 8. *Let C be large enough. Let $j=1,2,...k$, $c_j\in\mathbb{C}$, $|c_j|=1$, and $\alpha_j\in\mathbb{R}$. Let $A:=\{\alpha_j\}_{j=1}^k$. Suppose*

$$\int_{\mathbb{R}}(\sum_{\alpha\in A}\chi_{[\alpha-1,\alpha+1]}(x))^2\,dx\le S\,.\ \textit{Then}\ \int_0^1|\sum_{\alpha\in A}c_\alpha e^{i\alpha y}|^2\,dy\le C\,S\,.$$

Some key facts useful for its proof:

$$\int_0^1 |\sum_{\alpha\in A} c_\alpha e^{i\alpha\, y}\, dy|^2 \le e\int_0^\infty |\sum_{\alpha\in A} c_\alpha e^{i(\alpha+i)\, y}\, dy|^2 = e\int_{\mathbb{R}} \left|\sum_{\alpha\in A} \frac{c_\alpha}{\alpha+i-x}\right|^2 dx\,,$$

and the fact that $H^2(\mathbb{C}_+)$ is orthogonal to $\overline{H^2(\mathbb{C}_+)}$, so one can pass to the Poisson kernel.

4. The general case

Let us have k closed disjoint discs of radii $1/k$ located in the unit disc. We build k^n small discs of radii k^{-n} by iterating k linear maps from small discs onto the unit disc. Call the resulting union $S_k(n)$. We would like to show that exactly as in the case of $k=3$ considered above and in a very special case of $k=4$ considered in [4] $\mathrm{Fav}(S_k(n)) \le C\, n^{-c}$, $c>0$. However, presently we can prove only a weaker result.

Theorem 9.

$$Fav(S_k(n)) \le C\, e^{-c\,(\log n)^{1/2}},\ c>0\,.$$

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